MOD-2 EQUIVALENCE OF THE K-THEORETIC EULER AND SIGNATURE CLASSES

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This note proves that, as K-theory elements, the symbol classes of the de Rham operator and the signature operator on a closed manifold of even dimension are congruent mod 2. An equivariant generalization is given pertaining to the equivariant Euler characteristic and the multi-signature.

1. Introduction

It is well-known that the Euler characteristic $\chi(M)$ and the signature $\mathrm{Sign}(M)$ of a closed oriented manifold M of dimension $4n$ are two integers of the same parity. This fact is an easy consequence of Poincaré duality and we briefly recall its proof. Let $\beta_k = \dim H_k(M;\mathbb{R})$. By Poincaré duality, $\beta_k = \beta_{4n-k}$ and there is a non-degenerate bilinear form (the "intersection form") on $H_{2n}(M;\mathbb{R})$. Let β_{2n}^+ (respectively β_{2n}^-) be the dimension of a maximal subspace of $H_{2n}(M;\mathbb{R})$ on which the form is positive definite (respectively negative definite). By definition, $\mathrm{Sign}(M) = \beta_{2n}^+ - \beta_{2n}^-$. The following mod-2 congruence relation follows:

$$\chi(M) = \sum_{k=0}^{4n}(-1)^k\beta_k \equiv \beta_{2n}^+ + \beta_{2n}^- + \sum_{k=0}^{2n-1} 2\beta_k \equiv \beta_{2n}^+ - \beta_{2n}^- = \mathrm{Sign}(M).$$

We prove an analogous result on the level of the symbols (as K-theory classes) of the de Rham and signature operators on even-dimensional manifolds; we also give an equivariant generalization.

We first recall the construction of the two operators (cf. [2]).

Let M be a closed oriented smooth manifold of dimension $2n$. Equip M with a Riemannian metric. (The symbols, as K-theory classes, of the two operators will be independent of the choice of Riemannian metric and hence are smooth invariants. This is due to the fact that, for any metrics g_0 and g_1 and for $t \in [0,1]$, $tg_0 + (1-t)g_1$ is a Riemannian metric.) Let

$$\Omega^* = \Gamma(\Lambda^*(T^*M \otimes \mathbb{C})),$$

the space of smooth sections of the exterior algebra bundle $\Lambda^* = \Lambda^*(T^*_{\mathbb{C}}M)$ associated with the complexified cotangent bundle $T^*_{\mathbb{C}}M = T^*M \otimes \mathbb{C}$; i.e., Ω^* is the space of *complex* differential forms. Let

$$D = d + \delta : \Omega^* \to \Omega^*$$

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where d is the exterior derivative and δ its adjoint with respect to the Hermitian product on Ω^* induced by the Riemannian metric on M. The de Rham operator D^0 is defined to be the restriction of D to the subspace Ω^{even} of even-degree forms and thus takes value in the subspace Ω^{odd} of odd-degree forms:

$$D^0 = D|_{\Omega^{\text{even}}} : \Omega^{\text{even}} = \Gamma(\Lambda^{\text{even}}) \to \Omega^{\text{odd}} = \Gamma(\Lambda^{\text{odd}}).$$

The bundle map Hodge star $* : \Lambda^k \to \Lambda^{2n-k}$ is defined on each fiber $(\Lambda^k)_x$ by

$$\alpha \wedge *\beta =< \alpha, \beta > \cdot \operatorname{vol}(M)_x \quad \text{for } \alpha, \beta \in (\Lambda^k)_x$$

where $\operatorname{vol}(M) \in \Omega^{2n}$ is the volume form. The Hodge star has the property that

$$*(*\alpha) = (-1)^k \alpha \quad \text{for } \alpha \in \Lambda^k.$$

Define $\tau : \Lambda^k \to \Lambda^{2n-k}$ by letting

$$\tau = i^{n+k(k-1)} * .$$

It is easy to verify that τ is an involution on Λ^*. Then τ decomposes Λ^* into $\Lambda^+ \oplus \Lambda^-$, the $+1$ and -1 eigenbundles. The map τ induces an involution on Ω^* (which we also call τ) and decomposes Ω^* into $\Omega^+ \oplus \Omega^-$, where $\Omega^\pm$ are the (± 1)-eigenspaces. Note that $\Omega^\pm = \Gamma(\Lambda^\pm)$ and that D interchanges Ω^+ and Ω^- (since $D\tau = -\tau D$). The signature operator D^+ is defined to be the restriction of D to Ω^+:

$$D^+ = D|_{\Omega^+} : \Omega^+ = \Gamma(\Lambda^+) \to \Omega^- = \Gamma(\Lambda^-).$$

We now recall the K-theoretic symbol class associated with an elliptic differential operator. Suppose E_1 and E_2 are two complex vector bundles over M. Let $\pi : T^*_{\mathbb{C}}M \to M$ be the bundle projection. Associated with a differential operator $P : \Gamma(E_1) \to \Gamma(E_2)$, there is the (leading) symbol of P,

$$\sigma(P) : \pi^* E_1 \to \pi^* E_2.$$

$\sigma(P)$ is a bundle homomorphism. If, over the complement of the zero section of $T^*_{\mathbb{C}}M$, $\sigma(P)$ is a bundle isomorphism, P is then said to be *elliptic*. The symbol $\sigma(P)$ of an elliptic operator P determines a class $[\sigma(P)]$, the *K-theoretic symbol class of P*, in

$$K(T^*_{\mathbb{C}}M) = \widetilde{K}(D(T^*_{\mathbb{C}}M)/S(T^*_{\mathbb{C}}M)),$$

where $D(T^*_{\mathbb{C}}M)$ and $S(T^*_{\mathbb{C}}M)$ denote the closed unit disc bundle and the unit sphere bundle associated with $T^*_{\mathbb{C}}M$. We will review in more detail the transition from $\sigma(P)$ to $[\sigma(P)]$ in §2.

It is a standard fact that D^0 and D^+ are elliptic. We call $[\sigma(D^0)]$ the *K-theoretic Euler class of M* and $[\sigma(D^+)]$ the *K-theoretic signature class of M*. It is their relationship in the abelian group $K(T^*_{\mathbb{C}}M)$ that our Main Theorem pertains to.

Main Theorem. *If* $\dim M$ *is even, then* $[\sigma\,(D^0)] \equiv [\sigma\,(D^+)] \bmod 2K(T^*_{\mathbb{C}}M)$.

(Here, $2K(T^*_{\mathbb{C}}M)$ means $\{\alpha+\alpha : \alpha \in K(T^*_{\mathbb{C}}M)\}$.)

The Main Theorem implies, as a corollary, the mod-2 congruence between $\chi(M)$ and $\mathrm{Sign}(M)$. To see this, consider the index homomorphism $\mathrm{Index} : K(T^*_{\mathbb{C}}M) \to \mathbb{Z}$ and recall that $\mathrm{Index}([\sigma(D^0)]) = \chi(M)$ and $\mathrm{Index}([\sigma(D^+)]) = \mathrm{Sign}(M)$.

2. Preliminaries on K-theory

We review and establish some K-theoretic results.

Let (X, A) be a pair of connected compact Hausdorff spaces. Let E_1 and E_2 be two complex vector bundles over X with a bundle isomorphism $\sigma : E_1|_A \to E_2|_A$. The triple $(E_1, E_2; \sigma)$ determines a class $[E_1, E_2; \sigma]$ in $\widetilde{K}(X/A)$. We first recall its construction.

Let $X_i = X \times \{i\}$ and $A_i = A \times \{i\}$, $i = 1, 2$; let $Y = X_1 \cup_g X_2$, with $g : (a, 1) \mapsto (a, 2)$ for $a \in A$. (For notational convenience, we regard E_i as a bundle not only over X but also over X_1 and X_2.) We first construct bundles $E_{i,j}$'s over Y. To produce $E_{i,j}$, we glue $E_i|_{X_1}$ and $E_j|_{X_2}$ via $\varepsilon_{i,j} : E_i|_{A_1} \to E_j|_{A_2}$ where $\varepsilon_{1,2} = \sigma$, $\varepsilon_{2,1} = \sigma^{-1}$, and $\varepsilon_{i,i} = \mathrm{Id}_{E_i|_A}$.

Consider $E_{1,2} - E_{2,2} \in \widetilde{K}(Y)$. Evidently, its restriction to X_2 is $0 \in \widetilde{K}(X_2)$. As X_2 is a retract of Y and $X/A \cong Y/X_2$, the long exact sequence of K-theory for the pair (Y, X_2) yields the following short exact sequence:

$$0 \to \widetilde{K}(X/A) \to \widetilde{K}(Y) \to \widetilde{K}(X_2) \to 0.$$

Hence, $E_{1,2} - E_{2,2}$ is the image of a unique element in $\widetilde{K}(X/A)$, which we name $[E_1, E_2; \sigma]$. It can be shown that this element is invariant under variation of σ within its own homotopy class of bundle isomorphisms; see [1].

Lemma 2.1. *In the above notation, we have the following identities in* $\widetilde{K}(X/A)$*:*

1. $[E_1, E_2; \sigma] + [E_1', E_2'; \sigma'] = [E_1 \oplus E_1', E_2 \oplus E_2'; \sigma \oplus \sigma']$.
2. $[E_1, E_2; \sigma] + [E_2, E_3; \rho] = [E_1, E_3; \rho \circ \sigma]$.
3. $[E_2, E_1; \sigma^{-1}] = -\,[E_1, E_2; \sigma]$.

Proof: Part 1 is a direct consequence of the definition.

For Part 2, note that $\sigma \oplus \rho : E_1|_A \oplus E_2|_A \to E_2|_A \oplus E_3|_A$ is homotopic through bundle isomorphisms to

$$\varphi = \begin{pmatrix} 0 & -\operatorname{Id} \\ \rho \circ \sigma & 0 \end{pmatrix} : E_1|_A \oplus E_2|_A \to E_2|_A \oplus E_3|_A$$

via

$$t \mapsto \begin{pmatrix} \operatorname{Id} & 0 \\ t\rho & \operatorname{Id} \end{pmatrix} \begin{pmatrix} \operatorname{Id} & -t\rho^{-1} \\ 0 & \operatorname{Id} \end{pmatrix} \begin{pmatrix} \operatorname{Id} & 0 \\ t\rho & \operatorname{Id} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix}, \quad t \in [0,1].$$

Thus

$$\begin{aligned} [E_1, E_2; \sigma] + [E_2, E_3; \rho] &= [E_1 \oplus E_2, E_2 \oplus E_3; \sigma \oplus \rho] \quad \text{(by Part 1)} \\ &= [E_1 \oplus E_2, E_2 \oplus E_3; \varphi] \quad (\text{by } \varphi \simeq (\sigma \oplus \rho)) \\ &= [E_1 \oplus E_2, E_3 \oplus E_2; (\rho \circ \sigma) \oplus (-\operatorname{Id})] \quad (\text{by the formula for } \varphi) \\ &= [E_1, E_3; \rho \circ \sigma] + [E_2, E_2; (-\operatorname{Id})] \quad \text{(by Part 1)} \\ &= [E_1, E_3; \rho \circ \sigma] + [E_2, E_2; \operatorname{Id}] \end{aligned}$$

where the last equality follows from the fact that Id and $-\operatorname{Id}$ are homotopic through isomorphisms of *complex* vector bundles. It is trivial that $[E_2, E_2; \operatorname{Id}] = 0$ (which can be checked in $\widetilde{K}(Y)$). Part 2 then follows.

Part 3 follows directly from Part 2. □

Given an isomorphism of complex vector bundles $\sigma : E_1 \to E_2$, let $\sigma^* : E_2^* \to E_1^*$ denote the dual isomorphism; if E_1 and E_2 are equipped with Hermitian metrics, let $\widehat{\sigma} : E_2 \to E_1$ denote the adjoint bundle map. (Given Hermitian vector spaces V and W and a complex linear map $g : V \to W$, the adjoint $\widehat{g} : W \to V$ is defined by $\langle gv, w \rangle = \langle v, \widehat{g}w \rangle$ for $v \in V$ and $w \in W$. The adjoint of a bundle map can be fiberwise defined.) Since any complex linear map $g : V \to W$ remains complex linear when viewed as a map $g : \overline{V} \to \overline{W}$ where $\overline{V}$ and $\overline{W}$ are the conjugates V and W, we may view $\widehat{g}$ as $\widehat{g} : \overline{W} \to \overline{V}$. In the same way, we may view $\widehat{\sigma}$ as $\widehat{\sigma} : \overline{E_2} \to \overline{E_1}$.

Lemma 2.2. *Suppose that E_1 and E_2 are complex vector bundles equipped with Hermitian metrics and that $\sigma : E_1|_A \to E_2|_A$ is an isomorphism of complex vector bundles. Then, in $\widetilde{K}(X/A)$, we have:*

1. $[\overline{E_2}, \overline{E_1}; \widehat{\sigma}] = [E_2^*, E_1^*; \sigma^*]$.
2. $[E_1, E_2; \sigma]^* = \left[E_1^*, E_2^*; (\sigma^*)^{-1}\right]$.
3. $[E_2, E_1; \widehat{\sigma}] = -[E_1, E_2; \sigma]$.

Proof: For a complex linear map $g : V \to W$ between finite-dimensional Hermitian vector spaces, we have the following commutative diagram

$$\begin{array}{ccc} \overline{W} & \xrightarrow{\widehat{g}} & \overline{V} \\ \downarrow & & \downarrow \\ W^* & \xrightarrow{g^*} & V^* \end{array}$$

where the vertical maps are the *complex linear* isomorphisms

$$w \mapsto (\langle -, w\rangle : W \to \mathbb{R}) \in W^* \text{ and } v \mapsto (\langle -, v\rangle : V \to \mathbb{R}) \in V^*.$$

Part 1 then follows from the corresponding diagram of isomorphisms of bundles over A.

For Part 2, recall that the bundle $E_{i,j}$ over Y is given by gluing the bundle E_i over X_1 with E_j over X_2 via a bundle isomorphism $\varepsilon_{i,j} : E_i|_{A_1} \to E_j|_{A_2}$. It follows that the bundle $E^*_{i,j}$ is formed by gluing E^*_i with E^*_j via $\varepsilon^*_{i,j} : E^*_j|_{A_2} \to E^*_i|_{A_2}$, or equivalently via $(\varepsilon^*_{i,j})^{-1} : E^*_i|_{A_1} \to E^*_j|_{A_2}$. Part 2 readily follows.

Part 3 follows from a string of equalities:

$$\begin{aligned} [E_2, E_1; \widehat{\sigma}] &= [(\overline{E_2})^*, (\overline{E_1})^*; \sigma^*] \quad \text{(by Part 1)} \\ &= \left[(\overline{E_2})^*, (\overline{E_1})^*; \left((\sigma^{-1})^*\right)^{-1}\right] \\ &= \left[\overline{E_2}, \overline{E_1}; \sigma^{-1}\right]^* \quad \text{(by Part 2)} \\ &= -\left[\overline{E_1}, \overline{E_2}; \sigma\right]^* \quad \text{(by Part 3 of Lemma 2.1)} \\ &= -\left(\overline{[E_1, E_2; \sigma]}\right)^* \\ &= -[E_1, E_2; \sigma] \end{aligned}$$

where the last equality follows from the bundle equivalence between the conjugate of a complex vector bundle and its dual. $\square$

3. Proof of the Main Theorem

We now collect the preliminary results to prove our Main Theorem.

Proof of the Main Theorem: Let M be an even-dimensional closed smooth manifold.

Because $\dim M$ is even, Λ^{even} and Λ^{odd} are both invariant under the involution τ. Thus, τ decomposes Λ^{even} and Λ^{odd} into (± 1)-eigenbundles, resulting in the following decomposition of Λ^*:

$$\begin{aligned} \Lambda^* &= \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}} \\ &= (\Lambda^{\text{even},+} \oplus \Lambda^{\text{even},-}) \oplus (\Lambda^{\text{odd},-} \oplus \Lambda^{\text{odd},+}). \end{aligned}$$

Likewise, Ω^{even} and Ω^{odd} can be decomposed into (± 1)-eigenspaces of τ, resulting in the following decomposition of Ω^*:

$$\begin{aligned}\Omega^* &= \Omega^{\text{even}} \oplus \Omega^{\text{odd}} \\ &= (\Omega^{\text{even},+} \oplus \Omega^{\text{even},-}) \oplus (\Omega^{\text{odd},-} \oplus \Omega^{\text{odd},+}).\end{aligned} \tag{1}$$

Then, D^0 can be diagonalized as follows:

$$D^0 = D^{0,+} \oplus D^{0,-} : \Omega^{\text{even},+} \oplus \Omega^{\text{even},-} \longrightarrow \Omega^{\text{odd},-} \oplus \Omega^{\text{odd},+}.$$

Regrouping the summands in (1), we have

$$\Omega^* = (\Omega^{\text{even},+} \oplus \Omega^{\text{odd},+}) \oplus (\Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}). \tag{2}$$

Clearly,

$$(\Omega^{\text{even},+} \oplus \Omega^{\text{odd},+}) \subset \Omega^+ \quad \text{and} \quad (\Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}) \subset \Omega^- \ .$$

Since $\Omega^* = \Omega^+ \oplus \Omega^-$, the decomposition (2) implies

$$\Omega^+ = \Omega^{\text{even},+} \oplus \Omega^{\text{odd},+} \quad \text{and} \quad \Omega^- = \Omega^{\text{odd},-} \oplus \Omega^{\text{even},-} \ .$$

Then, D^+ can be diagonalized as follows:

$$D^+ = D^{+,0} \oplus D^{+,1} : \Omega^{\text{even},+} \oplus \Omega^{\text{odd},+} \longrightarrow \Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}.$$

Since $\Omega^{\text{even/odd},\pm} = \Gamma(\Lambda^{\text{even/odd},\pm})$, we have

$$\begin{aligned} &\sigma(D^0) = \sigma(D^{0,+}) \oplus \sigma(D^{0,-}) : \pi^*\Lambda^{\text{even},+} \oplus \pi^*\Lambda^{\text{even},-} \longrightarrow \pi^*\Lambda^{\text{odd},-} \oplus \pi^*\Lambda^{\text{odd},+} \ , \\ \text{and} \quad &\sigma(D^+) = \sigma(D^{+,0}) + \sigma(D^{+,1}) : \pi^*\Lambda^{\text{even},+} \oplus \pi^*\Lambda^{\text{odd},+} \longrightarrow \pi^*\Lambda^{\text{odd},-} \oplus \pi^*\Lambda^{\text{even},-} \ .\end{aligned}$$

By Part 1 of Lemma 2.1,

$$\big[\sigma(D^0)\big] = \big[\sigma(D^{0,+})\big] + \big[\sigma(D^{0,-})\big] \quad \text{and} \quad \big[\sigma(D^+)\big] = \big[\sigma(D^{+,0})\big] + \big[\sigma(D^{+,1})\big] \, .$$

It is a matter of definition that

$$D^{0,+} = D^{+,0} \quad \text{and} \quad D^{0,-} = \widehat{D^{+,1}}.$$

Thus,

$$\sigma(D^{0,+}) = \sigma(D^{+,0}) \quad \text{and} \quad \sigma(D^{0,-}) = \sigma(\widehat{D^{+,1}}) = \widehat{\sigma(D^{+,1}}).$$

Therefore

$$\big[\sigma(D^{0,+})\big] = \big[\sigma(D^{+,0})\big]$$

and, by Part 3 of Lemma 2.2,

$$\left[\sigma(D^{0,-})\right] = \left[\widehat{\sigma(D^{+,1})}\right] = -\left[\sigma(D^{+,1})\right].$$

In summary,

$$\begin{aligned}\left[\sigma(D^0)\right] + \left[\sigma(D^+)\right] &= \left[\sigma(D^{0,+})\right] + \left[\sigma(D^{0,-})\right] + \left[\sigma(D^{+,0})\right] + \left[\sigma(D^{+,1})\right] \\ &= 2\left[\sigma(D^{0,+})\right]\end{aligned}$$

□

4. Discussion

By the K-theoretic Thom isomorphism theorem for complex vector bundles, $K(T^*_{\mathbb{C}}M)$ is a free $K(M)$-module of rank 1 generated by the K-theory Thom class. In this way, the Main Theorem admits interpretation in $K(M)$. The K-theory Thom class of $T^*_{\mathbb{C}}M \to M$ is simply the element

$$[\pi^*\Lambda^{\text{even}}, \pi^*\Lambda^{\text{odd}}; \mu] \in K(T^*_{\mathbb{C}}M)$$

with

$$\mu(v,\omega) = v \wedge \omega - \iota_v\omega \quad \text{for } v \in (T^*_{\mathbb{C}}M)_x \text{ and } \omega \in (\Lambda^{\text{even}})_x$$

where $\iota_v : \Lambda^k \to \Lambda^{k-1}$ is the interior multiplication by v. This is precisely the K-theoretic Euler class, i.e., the symbol class of the de Rham operator $[\sigma(D^0)] \in K(T^*_{\mathbb{C}}M)$, which corresponds (via Thom isomorphism) to $1 \in K(M)$. Thus, letting $S \in K(M)$ denote the element corresponding (via the Thom isomorphism) to $[\sigma(D^+)] \in K(T^*_{\mathbb{C}}M)$ and translating the Main Theorem into a statement in $K(M)$, we have that $S = 1 + 2x$ for some $x \in K(M)$.

When M admits an orientation-preserving smooth action by a compact Lie group G, the Main Theorem admits an equivariant generalization. (This is due to a suggestion by Sylvain Cappell.) Equip M with a G-invariant metric, i.e., a metric with respect to which G acts isometrically. Then, $[\sigma(D^0)]$ and $[\sigma(D^+)]$ can both be interpreted as G-equivariant K-theory classes, i.e., elements of $K_G(T^*_{\mathbb{C}}M)$, and the proof for the Main Theorem can be adapted to show the following.

Theorem. *Suppose that* $\dim M$ *is even and that a compact Lie group* G *acts smoothly on* M *preserving orientation. Then* $[\sigma(D^0)] \equiv [\sigma(D^+)] \bmod 2K_G(T^*_{\mathbb{C}}M)$.

When G is finite, we may apply G-Index : $K_G(T^*_{\mathbb{C}}M) \to R(G)$ to deduce from this result the mod-2 equivalence of the equivariant Euler characteristic and multi-signature (as virtual complex representations of G).

In closing, we mention a few related works. One is [4], which shows that the K-homology class of the de Rham operator is trivial; another is [3], which discusses the equivariant KO-theoretic Euler characteristic. A more recent work, [5], shows among other results that the mod-8 reduction of the K-homology class of the signature operator is an oriented homotopy invariant.

References

[1] M. F. Atiyah, *K-Theory* (Addison-Wesley, 1988).

[2] M. F. Atiyah and I. M. Singer, 'The Index of Elliptic Operators III', *Ann. of Math.* **87** (1968), 546–604.

[3] W. Lück and J. Rosenberg, 'Equivariant Euler characteristics and K-homology Euler classes for proper cocompact G-manifolds', *Geometry and Topology* **7** (2003), 569-613.

[4] J. Rosenberg, 'The K-homology Class of the Euler Characteristic Operator Is Trivial', *Proc. Amer. Math. Soc.* **127** (1999), 3467–3474.

[5] J. Rosenberg and S. Weinberger, 'The Signature Operator at 2', *Topology* **45** (2006), no. 1, 47–63.

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